Numer Algor (2017) 76:125–150
DOI 10.1007/s11075-016-0247-z

ORIGINAL PAPER

Numerical schemes for integro-differential equations with Erdélyi-Kober fractional operator

Łukasz Płociniczak[1] · Szymon Sobieszek[1]

Received: 29 May 2016 / Accepted: 30 November 2016 / Published online: 19 December 2016

Abstract This work investigates several discretizations of the Erdélyi-Kober fractional operator and their use in integro-differential equations. We propose two methods of discretizing E-K operator and prove their errors asymptotic behaviour for several different variants of each discretization. We also determine the exact form of error constants. Next, we construct a finite-difference scheme based on a trapezoidal rule to solve a general first order integro-differential equation. As is known from the theory of Abel integral equations, the rate of convergence of any finite-different method depends on the severity of kernel's singularity. We confirm these results in the E-K case and illustrate our considerations with numerical examples.

Keywords Erdelyi-Kober operator · Fractional calculus · Finite difference · Integro-differential equation

1 Introduction

Fractional calculus constitutes a very vast area in which many interesting mathematical and physical objects reside. From the point of view of the latter, fractional models many times happen to describe natural phenomena with incredible accuracy probably thanks to its intrinsic nonlocal properties [21, 38]. These, in turn, can be used to model history of the considered process and a variety of memory effects [36, 46]. There are many examples of applications of fractional models [5, 21]. One of the

✉ Łukasz Płociniczak
lukasz.plociniczak@pwr.edu.pl

[1] Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50370 Wrocław, Poland

most successful is anomalous diffusion [34, 36, 37], which can be observed in variety of situations such as moisture percolation in porous media [39], protein random walks in cells [53], telomere motion [7, 23] and diffusion of cosmic rays across the magnetic fields [11]. When considering self-similar solutions to a sub-diffusive evolution equation [19, 47], the fractional derivative operator (either Riemann-Liouville or Caputo) becomes the so-called Erdélyi-Kober (E-K) fractional integral [15, 27] which possesses many interesting mathematical and physical features [20, 41, 50].

The E-K operator, which we will denote by $I_{a,b,c}$, is weakly-singular and can be viewed as one of Volterra (or Abel) type. Its precise definition will be given in Section 2 but note that the general theory can be applied to it. However, utilizing specific features of E-K operator leads to many interesting results. A thorough exposition concerning the theory of E-K fractional integral is presented in the book [25]. In [1, 26], a number of solutions to the E-K integral equations have been obtained (but see also [33]) while in [24] further results for hyper-Bessel operator were given. Moreover, some questions about existence and uniqueness were answered in [22, 52].

There is a very abundant literature about numerical methods for both fractional differential equations and integral equations. To state only a few, we start from mentioning two classic monographs concerning numerical methods for Volterra (and Abel) integral equations [8, 31]. Also, the reader will find there a thorough treatment of integro-differential equations with Volterra operators. A more modern review paper [4] summarizes recent results on a variety of numerical ways of solving considered equations. As being inegro-differential operators, fractional derivatives can be treated similarly to the more general cases. However, certain advantages can be gained from exploiting particular structure of these operators. A review of numerical methods (as well as analytical results) for ordinary fractional differential equations has been given in [6], where a modern overview and practical algorithms are given. The book contains a number of interesting references to which interested reader is referred to. Also, the paper [16] discusses some general methods for ordinary fractional differential equations while [29] gives a analysis of a non-uniform grid approximation. Lastly, we would like to mention several papers discussing numerical approaches to time-fractional diffusion. A very thorough treatment has been given in [30] where a combined space-time spectral method was used. A similar setting of finite differences was also applied in [31]. Some recent works on the nonlinear case include papers on finite difference schemes for inverse problem [13] and single-phase flow in porous media [3].

The motivation behind this paper is a self-similar solution of time-fractional porous medium equation (see [40, 51]). As we noted before, E-K operator appears in such a situation very naturally as a part of ordinary integro-differential equation modelling moisture distribution in a variety of building materials [14, 28] (also see some new experimental results [55]). In our preceding works [42–45], we have devised a systematic way of approximating the solution of that equation by a simple, analytical formulas. It was then compared with the numerical solution to verify its applicability and accuracy. We noticed that the finite difference scheme for the time-fractional partial differential equation was very demanding on the computer power and obtaining an array of solutions for different values of was not practical. Nonlocality and nonlinearity of the investigated equation is the obvious reason for such a case. This paper is a first step in deriving a more optimal numerical method which is constructed for the self-similar ordinary rather than original partial differential equation. In what follows, we introduce two types of discretization of the E-K operator, find theirs truncation errors with exact error constants and apply those results to construct a second-order finite difference scheme which approximates the solution of the first order integro-differential equation with E-K operator $I_{a,b,c}$, namely

$$y' = f(x, y, I_{a,b,c}y). \tag{1}$$

The objective for future work will be to extend these results to the self-similar nonlinear time-fractional diffusion.

2 Discretization of the Erdélyi-Kober operator

Let us define the Erdélyi-Kober (E-K) fractional integral operator by the formula

$$I_{a,b,c}y(x) := \frac{1}{\Gamma(b)}\int_0^1 (1-s)^{b-1}s^a y\left(s^{1/c}x\right) ds, \quad x \in (0, X), \tag{2}$$

where y is at least locally integrable. The above definition is one of the few equivalent ones found in the literature. Others can be obtained by a change of the variable [26]. The definition that will be particularly useful for numerical calculations arises from the transformation $t = s^{1/c}x$ made in the (2). This leads to the Volterra operator representation

$$I_{a,b,c}y(x) = \frac{cx^{-c(a+b)}}{\Gamma(b)}\int_0^x (x^c - t^c)^{b-1}t^{c(a+1)-1}y(t)dt. \tag{3}$$

Although the form of the above integral looks more formidable than (2), it turns out that is possesses more pleasant numerical properties. Additionally, it is just a matter of simple calculation to show that the power-type expressions are eigenfunctions of E-K operator

$$I_{a,b,c}x^{\gamma} = \frac{\Gamma\left(a + \frac{\gamma}{c} + 1\right)}{\Gamma\left(a + b + \frac{\gamma}{c} + 1\right)}x^{\gamma}. \tag{4}$$

As for a, b and c we assume that

$$a > -1, \quad b > 0, \quad c > 0. \tag{5}$$

We will take the above assumption to be valid for the rest of our work unless differently stated. This specific choice of domains for a and b is required for the integral (2) to be convergent. However, by the analytic continuation, a and b can be assumed to lie within the domain of Beta function but we will not pursue this route here (but see [43]). Note also, that in some important applications, we have $a = 0$. Apart from that, as can be seen from the self-similar analysis of the time-anomalous diffusion equation (see [9, 12, 42]), the particular version of the E-K operator that arises there requires $c < 0$. However, we defer the analysis of such case to our future work and in the present paper we assume that $c > 0$. Additional results concerning self-similar solutions of the fractional differential equations and E-K operators can be found for example in [10, 17, 18, 48].

The main idea behind discretization of E-K operator is to apply a quadrature rule for approximating *only* the function y and not the rest of the integrand. This will allow us to conduct a part of calculations analytically minimizing the discretization error. The type of quadrature can be chosen according to be suited for a particular application (or preference) and here we consider rectangular, mid-point and trapezoidal quadratures. This overall procedure, throughout the literature, is called *product integration method* (see [32]).

First, fix x and consider the representation (2). Introduce a grid of the $[0, 1]$ interval

$$0 = s_0 < s_1 < s_2 < \cdots < s_i < \cdots < s_n = 1, \tag{6}$$

where $\max_i (s_{i+1} - s_i) \to 0$ as the grid is refined, i.e. $n \to \infty$. At this point, it is not necessary to discretize the x variable. Now, we have

$$I_{a,b,c}y(x) = \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a y\left(s^{1/c}x\right) ds, \tag{7}$$

and we consider several ways of approximating y on a subinterval $[s_i, s_{i+1})$. More specifically, we apply a chosen quadrature to the function $Y(s) := y(s^{1/c}x)$ for fixed x and c.

- **Rectangular rule.** Here, on each subinterval we build an approximating rectangle with its height equal to $Y(s_i)$. By $L^r_{a,b,c}$ denote the operator which gives the discretization of $I_{a,b,c}$. It has the form

$$L^r_{a,b,c}y(x) := \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} y(s_i^{1/c}x) \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a ds = \sum_{i=0}^{n-1} v_i^r(a,b)\, y(s_i^{1/c}x), \tag{8}$$

where we have defined the weights

$$v_i^r(a,b) := \frac{1}{\Gamma(b)} \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a ds = \frac{B(s_{i+1}; a+1, b) - B(s_i; a+1, b)}{\Gamma(b)}, \tag{9}$$

by use of the Incomplete Beta Function. As both Gamma and Beta functions are readily and optimally implemented in many popular scientific software packages, we almost never need to compute the integral in (9). The important special case, $a = 0$, can be evaluated explicitly

$$v_i^r(0,b) = \frac{(1-s_i)^b - (1-s_{i+1})^b}{\Gamma(b+1)}. \tag{10}$$

- **Mid-point rule.** Here, the height of the approximating rectangle is $Y(s_{i+1/2})$, where $s_{i+1/2} := (s_{i+1} + s_i)/2$. The mid-point rule discretization is as follows

$$L^m_{a,b,c}y(x) := \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} y(s_{i+1/2}^{1/c}x) \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a ds = \sum_{i=0}^{n-1} v_i^r(a,b)\, y(s_{i+1/2}^{1/c}x), \tag{11}$$

where the weights $v^r(a,b)$ are the same as in the rectangular rule.

- **Trapezoidal rule.** In the trapezoidal rule, we approximate the function Y by the line segments, i.e. $Y(s) \approx \left[(Y(s_{i+1}) - Y(s_i)) \,/\, (s_{i+1} - s_i) \right] (s - s_i) + Y(s_i)$ on $[s_i, s_{i+1})$. This gives us the discretization

$$\begin{aligned} L^t_{a,b,c} y(x) :&= \tfrac{1}{\Gamma(b)} \sum_{i=0}^{n-1} \tfrac{y(s_{i+1}^{1/c}x) - y(s_i^{1/c}x)}{s_{i+1}-s_i} \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a (s - s_i) ds \\ &\quad + y(s_i^{1/c}x) \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a ds \\ &= \sum_{i=0}^{n} v_i^t(a,b)\ y(s_i^{1/c}x), \end{aligned} \tag{12}$$

where the trapezoidal weights are defined by

$$v_i^t(a,b) := \begin{cases} B_0, & i = 0; \\ A_{i-1} + B_i, & 0 < i < n; \\ A_{n-1}, & i = n, \end{cases} \tag{13}$$

$$\begin{aligned} A_i &:= \tfrac{1}{\Gamma(b)} \tfrac{\delta_i B(a+2,b) - s_i\ \delta_i B(a+1,b)}{s_{i+1}-s_i}, \\ B_i &:= \tfrac{1}{\Gamma(b)} \left(\delta_i B(a+1,b) - \tfrac{\delta_i B(a+2,b) - s_i\ \delta_i B(a+1,b)}{s_{i+1}-s_i} \right), \end{aligned} \tag{14}$$

where we defined

$$\delta_i B(a,b) := B(s_{i+1}; a, b) - B(s_i; a, b). \tag{15}$$

Note that all of the above discretizations need to evaluate y at a point $s_i^{1/c}x$, which depends on the parameter c and the $[0, 1]$ grid. This is a drawback of the method since given the x-grid it would require additional approximation by interpolating values of y: $s_i^{1/c}x$ does not have to belong to the x-grid.

As it will become clear, more sensible in most situations is to consider the Volterra representation of the E-K operator (3). Here, we fix x and define the grid

$$0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_n = x. \tag{16}$$

Similarly as above we expand the E-K integral (3)

$$I_{a,b,c} y(x) = \frac{c x^{-c(a+b)}}{\Gamma(b)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (x^c - t^c)^{b-1} t^{c(a+1)-1} y(t) dt, \tag{17}$$

and apply the standard interpolations to the function y. This has the advantage that y will be calculated on the grid points.

- **Rectangular rule.** We approximate y on $[y_i, y_{i+1})$ by $y(t_i)$ and obtain

$$K^r_{a,b,c} y(x) := \frac{c x^{-c(a+b)}}{\Gamma(b)} \sum_{i=0}^{n-1} y(t_i) \int_{t_i}^{t_{i+1}} (x^c - t^c)^{b-1} t^{c(a+1)-1} dt. \tag{18}$$

Since we have moved the function y out of the integral, we are free to substitute back $s = (t/x)^c$ to get

$$K^r_{a,b,c} y(x) = \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} y(t_i) \int_{(t_i/x)^c}^{(t_{i+1}/x)^c} (1-s)^{b-1} s^a ds = \sum_{i=0}^{n-1} w_i^r(a,b,c) y(t_i), \tag{19}$$

where the weights

$$w_i^r(a,b,c) := \frac{B((t_{i+1}/x)^c;a+1,b) - B((t_i/x)^c;a+1,b)}{\Gamma(b)}, \tag{20}$$

differ from (9) only by points at which the Incomplete Beta Function is evaluated. The dependence on c has moved from the argument of y into the weight. The special case $a = 0$ is

$$w_i^r(0,b,c) = \frac{(1-(t_i/x)^c)^b - (1-(t_{i+1}/x)^c)^b}{\Gamma(b+1)}. \tag{21}$$

- **Mid-point rule.** Here, we simply have

$$K^m_{a,b,c}y(x) = \frac{1}{\Gamma(b)}\sum_{i=0}^{n-1} y(t_{i+1/2})\int_{(t_i/x)^c}^{(t_{i+1}/x)^c}(1-s)^{b-1}s^a ds = \sum_{i=0}^{n-1} w_i^r(a,b,c)y(t_{i+1/2}), \tag{22}$$

where the weights are the same as in the rectangular rule.

- **Trapezoidal rule.** We use the first-order Lagrange polynomial to approximate $y(x)$ for each interval $[t_i, t_{i+1}]$, i.e. $y(t) \approx \left[(y(t_{i+1}) - y(t_i))/(t_{i+1}-t_i)\right](t - t_i) + y(t_i)$ which gives us the trapezoidal quadrature

$$K^t_{a,b,c}y(x) = \sum_{i=0}^{n} w_i^t(a,b,c)y(t_i), \tag{23}$$

where the weights are defined by

$$w_i^t(a,b) := \begin{cases} D_0, & i = 0; \\ C_{i-1} + D_i, & 0 < i < n; \\ C_{n-1}, & i = n, \end{cases} \tag{24}$$

with

$$\begin{aligned} C_i &:= \tfrac{1}{\Gamma(b)} \tfrac{x\ \Delta_i B(a+1/c+1,b,c) - t_i\ \Delta_i B(a+1,b,c)}{t_{i+1}-t_i}, \\ D_i &:= \tfrac{1}{\Gamma(b)}\left(\Delta_i B(a+1,b,c) - \tfrac{x\ \Delta_i B(a+1/c+1,b,c) - t_i\ \Delta_i B(a+1,b,c)}{t_{i+1}-t_i}\right), \end{aligned} \tag{25}$$

where we defined

$$\Delta_i B(a,b,c) := B((t_{i+1}/x)^c;a,b) - B((t_i/x)^c;a,b). \tag{26}$$

Introducing the uniform grid, we can find the order of all above discretizations. First, however, we state two elementary lemmas concerning asymptotic behaviour of a occurring series.

Lemma 1 *For $a, b \in \mathbb{R}$ and $c > 0$ we have*

$$\frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^{a-1}\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1} \sim \begin{cases} \frac{1}{c}B\left(\frac{a}{c}, b\right), & a > 0 \text{ and } b > 0; \\ \ln n, & a = 0 \text{ and } b > 0; \\ c^{b-1}\ln n, & a > 0 \text{ and } b = 0; \\ \zeta(1-a)n^{-a}, & a < 0 \text{ and } b > a \\ c^{b-1}\zeta(1-b)n^{-b}, & b < 0 \text{ and } a > b \\ (1+c^{a-1})\zeta(1-a)n^{-a}, & a = b < 0, \end{cases} \tag{27}$$

as $n \to \infty$.

Proof First, consider the case $a, b > 0$. From the definition of Riemann sum, the following is a consequence of the integrability of $s^{a-1}(1-s^c)^{b-1}$ and a fact that for any $n \in \mathbb{N}$ the set $\{i/n : 1 \le i \le n-1\}$ is the partition of $(0, 1)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^{a-1}\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1} = \int_0^1 s^{a-1}(1-s^c)^{b-1}ds. \tag{28}$$

After substitution $t = s^c$ the last integral defines Beta function $c^{-1}B(ac^{-1}, b)$.

Now, assume that $a < 0$ and $b > a$. The sum (27) can then be rewritten as

$$n^{-a}\sum_{i=1}^{n-1}\frac{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1}}{i^{1-a}}, \tag{29}$$

where we have moved the largest power of n in front of the series. We have to show that the sum above converges to $\zeta(1-a)$. First, when $b \ge 1$ the sequence $(1 - (i/n)^c)^{b-1}$ is nondecreasing for any fixed i and thus bounded from above by 1 and by $1-(b-1)(i/n)^c$ from below. Hence, we can write

$$\sum_{i=1}^{n-1}\frac{1}{i^{1-a}} - \frac{b-1}{n^c}\sum_{i=1}^{n-1}\frac{1}{i^{1-a-c}} \le \sum_{i=1}^{n-1}\frac{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1}}{i^{1-a}} \le \sum_{i=1}^{n-1}\frac{1}{i^{1-a}}. \tag{30}$$

Notice that the majorizing series from above converges to $\zeta(1-a)$. The estimate from below has exactly the same limit and in order to see that we have to consider the magnitude of $c > 0$. More specifically, we have an elementary result which follows from the asymptotics of partial sums of the Riemann Zeta function (which can be shown by the Euler-Maclaurin formula)

$$\frac{1}{n^c}\sum_{i=1}^{n-1}\frac{1}{i^{1-a-c}} = \begin{cases} O(n^{-c}), & c+a < 0; \\ O(n^{-1}\ln n), & c+a = 0; \\ O(n^a), & c+a > 0 \end{cases} \quad \text{as } n \to \infty. \tag{31}$$

Since $a < 0$ and $c > 0$, all the above cases show that the whole expression goes to zero with $n \to \infty$. Therefore, from (30), we can infer that

$$\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1}}{i^{1-a}} = \sum_{i=1}^{\infty}\frac{1}{i^{1-a}} = \zeta(1-a). \tag{32}$$

Assume now that $a < b < 1$. Notice that the function $f_n(x) := (1-(x/n)^c)^{b-1}x^{a-1}$ for $0 \le x \le n$ has a minimum at $x_{min} = n(1+c(b-1)/(a-1))^{-1/c} \ge 0$. Define $i_{min} := [x_{min}]$ (where $[x]$ is the integral part of x) and decompose

$$\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1}}{i^{1-a}} = \lim_{n\to\infty}\left(\sum_{i=1}^{i_{min}}\frac{\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1}}{i^{1-a}} + \sum_{i=i_{min}+1}^{n-2}\frac{\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1}}{i^{1-a}} + \frac{\left(1-\left(1-\frac{1}{n}\right)^c\right)^{b-1}}{(n-1)^{1-a}}\right). \tag{33}$$

By the assumption $b > a$, the last term vanishes as $n \to \infty$ and we will show that the last but one has exactly the same limit. To this end, we bound the sum by an integral of the function f_n. Since f_n is nondecreasing for $i_{min} \le x \le n-2$ we have

$$\begin{aligned}\sum_{i=i_{min}+1}^{n-2}\frac{\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1}}{i^{1-a}} &\le \int_{i_{min}}^{n-2}\left(1-\left(\frac{1+x}{n}\right)^c\right)^{b-1}(1+x)^{a-1}dx\\ &= n^a\int_{(1+\frac{c(b-1)}{a-1})^{-1/c}+\frac{1}{n}}^{1-\frac{1}{n}}\left(1-y^c\right)^{b-1}y^{a-1}dy. \end{aligned}\tag{34}$$

We will show that the last integral converges to 0 as $n \to \infty$. Set $\epsilon := 1/n$ and use the L'Hospital's Rule

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{\epsilon^a}\int_{(1+\frac{c(b-1)}{a-1})^{-1/c}+\epsilon}^{1-\epsilon}(1-y^c)^{b-1}y^{a-1}dy\\ &\overset{H}{=}\lim_{\epsilon\to 0}\frac{-\epsilon^{b-1}(1-\epsilon)^{a-1}-\left(1-\left((1+\frac{c(b-1)}{a-1})^{-1/c}+\epsilon\right)^c\right)^{b-1}\left((1+\frac{c(b-1)}{a-1})^{-1/c}+\epsilon\right)^{a-1}}{a\epsilon^{a-1}} = 0,\end{aligned}\tag{35}$$

where the last equality is valid under our assumption $b > a$. We have thus shown that

$$\lim_{n\to\infty}\sum_{i=i_{min}+1}^{n-2}\frac{\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1}}{i^{1-a}} = 0. \tag{36}$$

To finally evaluate the limit in (33), we notice that

$$\sum_{i=1}^{i_{min}}\frac{\left(1-\frac{i}{n}\right)^{b-1}}{i^{1-a}} \le \left(1-\left(\frac{i_{min}}{n}\right)^c\right)^{b-1}\sum_{i=1}^{i_{min}}\frac{1}{i^{1-a}} \le \left(1-\frac{1}{1+\frac{c(b-1)}{a-1}}\right)^{b-1}\sum_{i=1}^{\infty}\frac{1}{i^{1-a}}, \tag{37}$$

which allows us to invoke the Lebesgue Dominated Convergence Theorem and obtain

$$\lim_{n\to\infty}\sum_{i=1}^{i_{min}}\frac{\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1}}{i^{1-a}} = \zeta(1-a). \tag{38}$$

Combining (38) and (36) with (33) proves the case of $a < b < 1$.

The inverted dependence $b < a$ follows the same reasoning with slight modifications. For example, when $b < 0$ and $a \geq 1$, the series (27) by a change of summation variable $i \to n - i$ can be written as

$$\frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)^{a-1}\left(1-\left(1-\frac{i}{n}\right)^{c}\right)^{b-1} \leq \frac{1}{n}\sum_{i=1}^{n-1}\left(1-\left(1-\frac{i}{n}\right)^{c}\right)^{b-1}, \tag{39}$$

where the inequality follows from the assumption $a \geq 1$. Now, since

$$\lim_{n\to\infty}\left(\frac{n}{i}\right)^{b-1}\left(1-\left(1-\frac{i}{n}\right)^{c}\right)^{b-1} = c^{b-1}, \tag{40}$$

we have

$$\frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)^{a-1}\left(1-\left(1-\frac{i}{n}\right)^{c}\right)^{b-1} \leq Cn^{-b}\sum_{i=1}^{n-1}\frac{1}{i^{1-b}}, \tag{41}$$

for some constant $C > 0$. Hence, Lebesgue Dominated Convergence Theorem can be applied giving

$$\begin{aligned}&\lim_{n\to\infty}\frac{n^b}{n}\sum_{i=1}^{n-1}\left(1-\tfrac{i}{n}\right)^{a-1}\left(1-\left(1-\tfrac{i}{n}\right)^{c}\right)^{b-1}\\ &\quad=\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{1}{i^{1-b}}\left(1-\tfrac{i}{n}\right)^{a-1}\left(\tfrac{n}{i}\right)^{b-1}\left(1-\left(1-\tfrac{i}{n}\right)^{c}\right)^{b-1} = c^{b-1}\zeta(1-b),\end{aligned} \tag{42}$$

by (40). The case with $b < a < 1$ follows the same line of reasoning as before and hence we omit the details (change the summation variable $i \to n - i$, bound the series by integral and apply Lebesgue's Theorem).

Assume now that $a = 0$ and $b > 0$ (the case with $b < 0$ is proved). Other case, i.e. $b = 0$ and $a > 0$ can be demonstrated in a similar way. The instance with $b \geq 1$ is a consequence of the Lebesgue's Theorem just as above (or elementary estimates). For $0 < b < 1$, we anticipate logarithmic asymptotic behaviour and to prove it use the bounds by appropriate integrals. Start with the estimate from below

$$\frac{1}{n}\sum_{i=1}^{n-1}\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1}\left(\frac{i}{n}\right)^{-1} \geq \sum_{i=1}^{n-1}\frac{1}{i}, \tag{43}$$

since $(1-(i/n)^c)^{b-1} \geq 1$. From above, we estimate by dividing the sum into two parts where the summand is monotone. Next, we bound each sum by the corresponding integral

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n-1}\left(1-\left(\tfrac{i}{n}\right)^{c}\right)^{b-1}\left(\tfrac{i}{n}\right)^{-1} \leq \int_1^{i_{min}}\left(1-\left(\tfrac{x}{n}\right)^{c}\right)^{b-1}x^{-1}dx\\ &\qquad+\int_{i_{min}}^{n-2}\left(1-\left(\tfrac{x+1}{n}\right)^{c}\right)^{b-1}(x+1)^{-1}\,dx+\frac{\left(1-\left(1-\frac{1}{n}\right)^{c}\right)^{b-1}}{n-1}.\end{aligned} \tag{44}$$

A change of variable $x = ny$ and $x + 1 = ny$ in each integral respectively lets us refine the estimate into

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n-1}\left(1-\left(\tfrac{i}{n}\right)^{c}\right)^{b-1}\left(\tfrac{i}{n}\right)^{-1} &\leq \int_{\frac{1}{n}}^{\frac{1}{2-b}}(1-y^{c})^{b-1}\,y^{-1}dy\\ &\quad+\int_{(1+c(1-b))^{-1/c}}^{1-\frac{1}{n}}(1-y^{c})^{b-1}\,y^{-1}dy+\frac{\left(1-\left(1-\frac{1}{n}\right)^{c}\right)^{b-1}}{n-1}.\end{aligned}\tag{45}$$

We thus can see that the second integral goes to a positive constant while the last term becomes zero as $n \to \infty$. It suffices to show that the first term above has logarithmic asymptotics. To this end use the L'Hospital's rule and obtain (we implicitly substitute $\epsilon = 1/n$ to conduct the differentiation)

$$\lim_{n\to\infty}\frac{1}{\ln n}\int_{\frac{1}{n}}^{(1+c(1-b))^{-1/c}}\left(1-y^{c}\right)^{b-1}y^{-1}dy \overset{H}{=} \lim_{n\to\infty}\frac{-\left(-\frac{1}{n^2}\right)\left(1-\left(\frac{1}{n}\right)^{c}\right)^{b-1}\left(\frac{1}{n}\right)^{-1}}{\frac{1}{n}}=1.\tag{46}$$

Combining (43), (45) with (46) lets us conclude that

$$\lim_{n\to\infty}\frac{1}{\ln n}\frac{1}{n}\sum_{i=1}^{n-1}\left(1-\left(\frac{i}{n}\right)^{c}\right)^{b-1}\left(\frac{i}{n}\right)^{-1}=1.\tag{47}$$

We are left with proving the case $a = b < 0$. For equal parameters, the function $f_n(x)$ defined above has its minimum at $x_{min} = n(1 + c)^{-1/c}$. Setting $i_{min} = [x_{min}]$ and decomposing our sum yields

$$\sum_{i=1}^{n-1}\frac{1}{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{1-a}i^{1-a}}=\sum_{i=1}^{i_{min}}\frac{1}{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{1-a}i^{1-a}}+\sum_{i=i_{min}+1}^{n-1}\frac{1}{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{1-a}i^{1-a}}.\tag{48}$$

We change the summation index in the last term to $n - i$ to obtain

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{1}{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{1-a}i^{1-a}}&=\sum_{i=1}^{i_{min}}\frac{1}{\left(1-\left(\frac{i}{n}\right)^{c}\right)^{1-a}i^{1-a}}\\&\quad+\sum_{i=1}^{n-i_{min}-1}\frac{n^{a-1}}{\left(1-\left(1-\frac{i}{n}\right)^{c}\right)^{1-a}\left(1-\frac{i}{n}\right)^{1-a}}.\end{aligned}\tag{49}$$

Observe that $nc(1+c)^{-1/c} \leq i_{min}$ and $nc(1+c)^{-1/c} < n-i_{min}-1 \leq n/2-1$ hence by the same argument as before we can use the Lebesue Dominated Convergence Theorem and (40) to finally get

$$\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{1}{\left(1-\frac{i}{n}\right)^{1-a}i^{1-a}}=(1+c^{a-1})\zeta(1-a),\tag{50}$$

for $a = b < 0$. This concludes the proof. □

For further reference, it will be useful to define the following function

$$\gamma_{a,b,c}(s) := (1-s^c)^{b-1}s^a, \quad s \in (0,1), \quad a,b \in \mathbb{R}, \quad c > 0. \tag{51}$$

The next lemma is an auxiliary result giving another property of $\gamma_{a,b,c}$.

Lemma 2 *Fix $m \geq 1$. If $-1 < a < m-1$ or $0 < b < m$, we have*

$$T_n := \frac{1}{n^{k+m}} \sum_{i=1}^{n-1} \gamma_{a,b,c}^{(m)}(\sigma_i) = \begin{cases} O\left(n^{-(k+m-1)}\right), & a > m-1 \text{ and } b > m; \\ O\left(n^{-(k+\delta-1)}\right), & -1 < a < m-1 \text{ or } 0 < b < m \end{cases} \quad \text{as} \quad n \to \infty, \tag{52}$$

where $\delta := \min\{a+1, b\}$ and $\sigma_i \in (\frac{i}{n}, \frac{i+1}{n})$. Moreover, it follows that $T_n = o(n^{-(k-1)})$ as $n \to \infty$.

Proof Notice that the mth derivative of $\gamma_{a,b,c}$ will contain terms of the form $(1-\sigma_i^c)^\alpha s^\beta$, where by inspection the lowest exponents will be $\alpha := b-m-1$ or $\beta := a-m$ (lowest exponents dominate the asymptotics). Each differentiation will bring one $c-1$ term into the β exponent but overall it will be larger than $a-m$ (since $c>0$). Hence, it suffices to consider only two cases.

If $a > m-1$ and $b > m$, then $\alpha > 0$ and $\beta > -1$ and thus by Lemma 1 (and the fact $i/n < \sigma_i < (i+1)/n$) it follows that $T_n = O(n^{-(k+m-1)})$, which for $m > 1$ is of smaller order than $n^{-(k-1)}$.

Now, assume that $a < m-1$ and $b > m$. Again, by Lemma 1 the dominant term is $O(n^{m-\delta+1})$, hence $T_n = O(n^{-(k+\delta-1)}$ as $n \to \infty$. Because $\delta > 0$, we have $k+\delta-1 > k-1$ therefore $T_n = o(n^{-(k-1)})$. Logarithmic behaviour is dealt with the same way. □

We can now find the orders of discretization errors of the approximation operators $L_{a,b,c}$ and $K_{a,b,c}$. We will limit our reasoning to the case of uniform grids, i.e. $s_i = i/n$ and $t_i = x\, i/n$, where n is the number of grid divisions. Note that $s_{i+1}-s_i = 1/n$ and $t_{i+1}-t_i = x/n$.

Theorem 1 (Discretization errors) *Fix $a, b, c > 0$ an assume that $y \in C^2(0,X)$. Then, for a fixed $x \in (0,X)$ the discretization errors corresponding to the operator $I_{a,b,c}$ have the following asymptotic behaviour as $n \to \infty$.*

- *Rectangular rule*

$$I_{a,b,c}y(x) - L^r_{a,b,c}y(x) \sim \frac{x}{c}\frac{y'(\sigma^{\frac{1}{c}}x)}{2\Gamma(b)} \begin{cases} n^{-1}B\left(\frac{1}{c}+a, b\right), & \frac{1}{c}+a > 0; \\ n^{-1}\ln n, & \frac{1}{c}+a = 0; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(1-\frac{1}{c}-a\right), & \frac{1}{c}+a < 0, \end{cases}$$

$$I_{a,b,c}y(x) - K^r_{a,b,c}y(x) \sim \frac{1}{n}\frac{x}{2\Gamma(b)}y'(\tau)B(a+1,b). \tag{53}$$

- *Trapezoidal rule*

$$I_{a,b,c}y(x)- \ L^t_{a,b,c}y(x) \sim \begin{cases} -\frac{x}{c}\left(\frac{1}{c}-1\right)\frac{y'(\sigma^{\frac{1}{c}}x)}{12\Gamma(b)}\begin{cases} n^{-2}B\left(\frac{1}{c}+a-1,b\right), & \frac{1}{c}+a>1; \\ n^{-2}\ln n, & \frac{1}{c}+a=1; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(2-\frac{1}{c}-a\right), & \frac{1}{c}+a<1, \end{cases} & c\neq 1; \\ -\frac{1}{n^2}\frac{x^2y''(\sigma x)}{12\Gamma(b)}B(a+1,b), & c=1. \end{cases} \tag{54}$$

$$I_{a,b,c}y(x) - K^t_{a,b,c}y(x) \sim -\frac{1}{n^2}\frac{x^2}{12\Gamma(b)}y''(\tau)B(a+1,b). \tag{55}$$

- *Midpoint rule*

$$I_{a,b,c}y(x) - L^m_{a,b,c}y(x) = \begin{cases} O(n^{-2}), & a+\frac{1}{c}>1 \text{ and } b\geq 1; \\ O(n^{-2}\ln n), & a+\frac{1}{c}=1 \text{ or } b\geq 1; \\ O(n^{-(1+\min\{b,a+\frac{1}{c}\})}), & -1<a+\frac{1}{c}<1 \text{ or} 0<b<1, \end{cases}$$

$$I_{a,b,c}y(x) - K^m_{a,b,c}y(x) = \begin{cases} O(n^{-2}), & c(a+1)\geq 1 \text{ and } b\geq 1; \\ O(n^{-(1+\min\{c(a+1),b\})}), & 0<c(a+1)<1 \text{ or} 0<b<1. \end{cases} \tag{56}$$

Here, $\sigma \in (0,1)$ and $\tau \in (0,x)$ depend on parameters a, b, c, function y and can be different for each discretization.

Remark 1 Note that it is possible that some of the above formulas can indicate that the difference between discretization and the E-K operator is asymptotic to zero. This simply means that the error is of higher order than stated. In other words, asymptotic relations above give the lowest order of discretization error. Finding the whole asymptotic expansion of these quantities is one of the objectives of our future work.

Proof Theorem 1 Let us start with the simplest case of the rectangular rule. First, consider the discretization of the first type, i.e. operator $L^r_{a,b,c}$ defined in (8). Expanding in the Taylor series we have for $s \in [s_i, s_{i+1})$ and some $\widetilde{\sigma} \in (s_i, s_{i+1})$

$$y(s^{\frac{1}{c}}x) = y(s_i^{\frac{1}{c}}x) + \frac{x}{c}\widetilde{\sigma}_i^{\frac{1}{c}-1}y'(\widetilde{\sigma}_i^{\frac{1}{c}}x)(s-s_i), \tag{57}$$

which allows us to write

$$I_{a,b,c}y(x) = L^r_{a,b,c}y(x) + R^r, \tag{58}$$

where

$$R^r = \frac{x}{c}\frac{1}{\Gamma(b)}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} \widetilde{\sigma}_i^{\frac{1}{c}-1}y'(\widetilde{\sigma}_i^{\frac{1}{c}}x)(1-s)^{b-1}s^a(s-s_i)ds. \tag{59}$$

Now, fix i and define an auxiliary function

$$F(z) := \int_{s_i}^{z}(1-s)^{b-1}s^a(s-s_i)ds. \tag{60}$$

Then, use the Mean Value Theorems, in turn for integrals and sums (notice that $(s - s_i)$ does not change its sign for $s \in (s_i, s_{i+1})$), to obtain

$$R^r = \frac{x}{c}\frac{y'(\sigma^{\frac{1}{c}}x)}{\Gamma(b)}\sum_{i=0}^{n-1}\widetilde{\sigma}_i^{\frac{1}{c}-1}F(s_{i+1}), \tag{61}$$

for some $\sigma \in (0, 1)$. We do not pursue here for explicitly evaluating the integral defining F (which can be done in terms of the Beta functions) but rather to retrieve its leading-order behaviour as $n \to \infty$. This will clearly indicate the order of discretization error. To proceed, we expand F in the Taylor series noting that $F(s_i) = F'(s_i) = 0$ which after setting $s = s_{i+1}$ for $1 \le i \le n-1$ gives

$$F(s_{i+1}) = \frac{1}{2}(1-s_i)^{b-1}s_i^a(s_{i+1}-s_i)^2+\frac{d^2}{ds^2}\left[(1-s)^{b-1}s^a(s-s_i)\right]_{s=\widehat{\sigma}_i}\frac{(s_{i+1}-s_i)^3}{6}, \tag{62}$$

for some intermediate point $\widehat{\sigma}_i \in (s_i, s_{i+1})$. Because $s_{i+1} - s_i = 1/n$ the first term in the above formula is $O(n^{-2})$ (by Lemma 1) and we have to show that the second is always of higher order. When we expand the derivative we obtain

$$\frac{d^2}{ds^2}\left[\gamma_{a,b,1}(s)(s-s_i)\right]_{s=\widehat{\sigma}_i}\frac{1}{6n^3} = \left(\gamma''_{a,b,1}(\widehat{\sigma}_i)(\widehat{\sigma}_i - s_i) + 2\gamma'_{a,b,1}(\widehat{\sigma}_i)\right)\frac{1}{6n^3}, \tag{63}$$

where, as before, $\gamma_{a,b,c}$ is defined in (51). Now, multiplying by $\widetilde{\sigma}_i^{\frac{1}{c}-1}$ and summing over $1 \le i \le n-1$ gives

$$\left|\frac{1}{6n^3}\sum_{i=1}^{n-1}\widetilde{\sigma}_i^{\frac{1}{c}-1}\frac{d^2}{ds^2}\left[\gamma_{a,b,1}(s)(s-s_i)\right]_{s=\widehat{\sigma}_i}\right| \le \frac{1}{6n^4}\sum_{i=1}^{n-1}\widetilde{\sigma}_i^{\frac{1}{c}-1}\left|\gamma''_{a,b,1}(\widehat{\sigma}_i)\right| + \frac{2}{6n^3}\sum_{i=1}^{n-1}\widetilde{\sigma}_i^{\frac{1}{c}-1}\left|\gamma'_{a,b,1}(\widehat{\sigma}_i)\right|. \tag{64}$$

Lemma 2 immediately states that the right-hand side is $o(n^{-1})$ (take $k = 2$ with $m = 2$ and $m = 1$ for the first and second sum respectively). From this estimate on, the remainder in (62) we can conclude

$$R^r \overset{n\to\infty}{\sim} \frac{x}{c}\frac{y'(\sigma^{\frac{1}{c}}x)}{2\Gamma(b)}\begin{cases} n^{-1}B\left(\frac{1}{c}+a, b\right), & \frac{1}{c}+a > 0; \\ n^{-1}\ln n, & \frac{1}{c}+a = 0; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(1-\frac{1}{c}-a\right), & \frac{1}{c}+a < 0, \end{cases} \tag{65}$$

where Lemma 1 has once again been used in determining the asymptotic form of the series (the $i = 0$ term in (61) vanishes due to the convergence of integral).

The discretization error for the second method (19) can be obtained in a similar way. In the second form of the operator $I_{a,b,c}$ (17) expand y at $t = t_i$ into the Taylor series to obtain

$$\begin{aligned} I_{a,b,c}y(x) &= K^r_{a,b,c}y(x)+\frac{cx^{-c(a+b)}}{\Gamma(b)}\sum_{i=0}^{n-1}y'(\widetilde{\tau}_i)\int_{t_i}^{t_{i+1}}(x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)dt \\ &=: K^r_{a,b,c}y(x) + P^r, \end{aligned} \tag{66}$$

where we have used the mean value theorem and defined the remainder P^r. Now, introduce the auxiliary function

$$G(z) := \int_{t_i}^{z} (x^c - t^c)^{b-1} t^{c(a+1)-1}(t - t_i)dt, \tag{67}$$

expand it into Taylor-Lagrange series at $t = t_i$ and evaluate at $t = t_{i+1}$

$$G(t_{i+1}) = (x^c - t_i^c)^{b-1} t_i^{c(a+1)-1} \frac{x^2}{2n^2} + \frac{d^2}{dt^2}\left[(x^c - t^c)^{b-1} t^{c(a+1)-1}(t - t_i)\right]|_{t=\widehat{\tau_i}} \frac{x^3}{6n^3}, \tag{68}$$

where we used $t_{i+1} - t_i = x/n$. Observe that we can write

$$\frac{d^2}{dt^2}\left[(x^c - t^c)^{b-1} t^{c(a+1)-1}(t - t_i)\right]|_{t=\widehat{\tau_i}} = x^{2(1-c)+c(b-a)} \frac{d^2}{dt^2}\left[\gamma_{c(a+1)-1,b,c}\left(\frac{t}{x}\right) \times \left(\frac{t}{x} - \frac{t_i}{x}\right)\right]|_{t=\widehat{\tau_i}}, \tag{69}$$

and $i/n \leq t/x < (i+1)/n$ hence by Lemma 2 it is easy to show that the second derivative term is of higher order than the first one. Inserting the above formula into definition of P^r we thus have

$$P^r \overset{n\to\infty}{\sim} \frac{1}{n} \frac{x}{2\Gamma(b)} y'(\tau) B(a+1, b). \tag{70}$$

In the same manner, we can obtain the discretization error for the trapezoidal operator L^t as defined in (12). Here, we use the well-known remainder form of the polynomial interpolation

$$I_{a,b,c}y(x) = L^t_{a,b,c}y(x) + R^t, \tag{71}$$

where

$$R^t = \frac{1}{2\Gamma(b)} \sum_{i=0}^{n-1} Y''(\widetilde{\sigma_i}) \int_{s_i}^{s_{i+1}} \gamma_{a,b,1}(s)(s - s_i)(s - s_{i+1})ds, \tag{72}$$

where $\widetilde{\sigma_i} \in (s_i, s_{i+1})$ and

$$Y''(s) = \frac{x}{c}\left(\frac{1}{c} - 1\right) s^{\frac{1}{c}-2} y'(s^{\frac{1}{c}}x) + \left(\frac{x}{c} s^{\frac{1}{c}-1}\right)^2 y''(s^{\frac{1}{c}}x). \tag{73}$$

The procedure of finding the leading-order term of the integral is the same as in the rectangular rule. Expanding the integral, we have

$$\int_{s_i}^{s_{i+1}} \gamma_{a,b,1}(s)(s - s_i)(s - s_{i+1})ds = -\gamma_{a,b,1}(s_i)\frac{1}{6n^3} + \gamma'_{a,b,1}(s_i)\frac{1}{6n^4} + \frac{d^3}{ds}\left[\gamma_{a,b,1}(s)(s - s_i)(s - s_{i+1})\right]|_{s=\widehat{\sigma_i}} \frac{1}{24n^4}. \tag{74}$$

Now, as before, we have to show that the third derivative term above multiplied by $\widetilde{\sigma_i}^{\frac{1}{c}-2}$ (by (72) and (73)) and summed over $1 \leq i \leq n-1$ is $o(n^{-2})$ as $n \to \infty$. The algebra is more cumbersome than in the rectangular case but the reasoning goes exactly the same way as in (64). Hence, we omit the details stating only the leading-order term which can be obtained by Lemma 1. The leading-order term for the case $c \neq 1$ is the following

$$R^t \overset{n\to\infty}{\sim} -\frac{x}{c}\left(\frac{1}{c}-1\right)\frac{y'(\sigma^{\frac{1}{c}}x)}{12\Gamma(b)}\begin{cases} n^{-2}B\left(\frac{1}{c}+a-1,b\right), & \frac{1}{c}+a>1; \\ n^{-2}\ln n, & \frac{1}{c}+a=1; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(2-\frac{1}{c}-a\right), & \frac{1}{c}+a<1, \end{cases} \tag{75}$$

while for $c=1$ it becomes

$$R^t \overset{n\to\infty}{\sim} -\frac{x^2 y''(\sigma x)}{12\Gamma(b)}B(a+1,b)\frac{1}{n^2}. \tag{76}$$

We quickly turn to the operator $K^t_{a,b,c}$ defined in (23). Reasoning as above, we have

$$I_{a,b,c}y(x) = K^t_{a,b,c}y(x) + P^t, \tag{77}$$

where

$$P^t := \frac{cx^{-c(a+b)}}{\Gamma(b)}\sum_{i=0}^{n-1} y''(\widetilde{\tau_i})\int_{t_i}^{t_{i+1}}(x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)(t-t_{i+1})dt. \tag{78}$$

When we expand the integral into the its Taylor series, we obtain

$$\begin{aligned} &\int_{t_i}^{t_{i+1}} (x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)(t-t_{i+1})dt = \\ &\quad -(x^c-t_i^c)^{b-1}t_i^{c(a+1)-1}\frac{x^3}{6n^3} + \frac{d}{dt}\left[(x^c-t^c)^{b-1}t^{c(a+1)-1}\right]_{t=t_i}\frac{x^4}{6n^4} \\ &\quad +\frac{d^3}{dt^3}\left[(x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)(t-t_{i+1})\right]|_{t=\widehat{\tau_i}}\frac{x^4}{24n^4}. \end{aligned} \tag{79}$$

And once again, factoring out constant x and invoking Lemma 2 gives us that the second and third terms are of higher order than the first which implies

$$P^t \overset{n\to\infty}{\sim} -\frac{1}{n^2}\frac{x^2}{12\Gamma(b)}y''(\tau)B(a+1,b). \tag{80}$$

Finally, we move to the midpoint rule which presents a slightly different case than the previous quadratures. For the operator $L^m_{a,b,c}$ defined in (11), we write a two-term expansion

$$Y(s) = Y(s_{i+1/2}) + Y'(s_{i+1/2})(s-s_{i+1/2}) + Y''(\widetilde{\sigma_i})\frac{(s-s_{i+1/2})^2}{2}, \tag{81}$$

whence by (11)

$$I_{a,b,c}y(x) = L^m_{a,b,c}y(x) + R^m, \tag{82}$$

where by the mean-value theorem

$$\begin{aligned} R^m &= \frac{1}{\Gamma(b)}\sum_{i=0}^{n-1} Y'(s_{i+1/2})\int_{s_i}^{s_{i+1}}(1-s)^{b-1}s^a(s-s_{i+1/2})ds \\ &\quad +\frac{1}{2}Y''(\widehat{\sigma_i})\int_{s_i}^{s_{i+1}}(1-s)^{b-1}s^a(s-s_{i+1/2})^2 ds. \end{aligned} \tag{83}$$

Since $s_{i+1/2} = (s_{i+1} + s_i)/2$ depends on $s_{i+1,}$ we have to be careful in expanding the integrals. To this end, fix i and define the two auxiliary functions

$$\begin{aligned} H_1(z) &:= \int_{s_i}^{z} (1-s)^{b-1} s^a \left(s - \tfrac{z+s_i}{2}\right) ds, \\ H_2(z) &:= \int_{s_i}^{z} (1-s)^{b-1} s^a \left(s - \tfrac{z+s_i}{2}\right)^2 ds. \end{aligned} \tag{84}$$

An expansion at $z = s_i$ evaluated at $z = s_{i+1}$ gives for $n \to \infty$

$$R^m \sim \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} Y'(s_{i+1/2}) \gamma'_{a,b,1}(\widetilde{\sigma_i}) \frac{(s_{i+1}-s_i)^3}{6} + Y''(\widehat{\sigma_i}) \gamma_{a,b,1}(\overline{\sigma_i}) \frac{(s_{i+1}-s_i)^3}{24}. \tag{85}$$

We see that the term multiplying Y'' is always of order higher or equal than the other one since by Lemma 2 $\gamma'_{a,b,1}$ can diminish the convergence. Also, $Y'(s) = C_1 s^{\frac{1}{c}-1}$ and $Y''(s) = C_2 s^{\frac{1}{c}-2} + C_3 s^{\frac{2}{c}-2}$, for some constants $C_{1,2,3}$. After plugging it in the above formula, we can collect the powers of s_i, $\widetilde{\sigma_i}$ or $\overline{\sigma_i}$ to obtain the dominant behaviour

$$R^m = \begin{cases} O(n^{-2}), & a + \frac{1}{c} > 1 \text{ and } b \geq 1; \\ O(n^{-2} \ln n), & a + \frac{1}{c} = 1 \text{ or } b \geq 1; \\ O(n^{-(1+\min\{b, a+\frac{1}{c}\})}), & -1 < a + \frac{1}{c} < 1 \text{ or } 0 < b < 1. \end{cases} \quad n \to \infty. \tag{86}$$

The reasoning for the second discretization of $I_{a,b,c}$, i.e. operator $K^m_{a,b,c}$ undergoes the same path yielding

$$P^m = \begin{cases} O(n^{-2}), & c(a+1) \geq 1 \text{ and } b \geq 1; \\ O(n^{-(1+\min\{c(a+1), b\})}), & 0 < c(a+1) < 1 \text{ or } 0 < b < 1. \end{cases} \quad \text{as } n \to \infty. \tag{87}$$

This ends the proof. □

We can see that, as anticipated, discretization done using L is always inferior to the operator K since it is sensitive to the value of c. This is especially transparent for rectangular and trapezoidal rules which loose the convergence rate for $c^{-1} + a < 0$ or $c^{-1} + a < 1$ respectively. Furthermore, the midpoint rule suffers from a singularity of the kernel even for the K discretization. That is, it looses its convergence rate if $0 < b < 1$. Nevertheless, the strongest advantage of this method lies in the unquestionable simplicity of implementation. More on the convergence properties of midpoint quadratures for weakly-singular kernels can be found in [35, 49] where, for example, the full asymptotic expansion of the error term has been found. The loss of convergence rate in numerical methods for weakly-singular Volterra (or Abel) integral equations is a known phenomenon [2, 54]. The analysis is interesting and reader is referred to [32] where a thorough exposition on numerical solution of Volterra and Abel integral equation is presented.

Note also that the real rate of convergence should depend on the operated function y and its derivatives. To quickly see this, we can take $y(x) = x^m$ for $m > 0$. Then in (61), instead of $1/c$ in the exponent, we would have m/c hence the kernel could loose its singularity giving better convergence rate. A thorough analysis of such cases is a subject of our future work.

Before we proceed to the numerical simulations, we can state a corollary to the Theorem 1. From the inspection of its proof, we can see that the continuity of derivatives of y implies boundedness on $[0, X]$. This allows us to write uniform bounds for the errors.

Corollary 1 *Fix $a, b, c > 0$ an assume that $y \in C^2(0, X)$. Let M denote the common bound for y' and y'', i.e. $|y'(x)| \leq M$ and $|y''(x)| \leq M$ for $x \in (0, X)$. Then, the following uniform bounds on the discretization errors take place.*

- *Rectangular rule*

$$\left|I_{a,b,c}y(x) - L^r_{a,b,c}y(x)\right| \leq \frac{X}{c}\frac{M}{2\Gamma(b)} \begin{cases} n^{-1}B\left(\frac{1}{c}+a, b\right), & \frac{1}{c}+a>0; \\ n^{-1}\ln n, & \frac{1}{c}+a=0; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(1-\frac{1}{c}-a\right), & \frac{1}{c}+a<0, \end{cases}$$

$$\left|I_{a,b,c}y(x) - K^r_{a,b,c}y(x)\right| \leq \frac{1}{n}\frac{X}{2\Gamma(b)}MB(a+1,b). \tag{88}$$

- *Trapezoidal rule*

$$|I_{a,b,c}y(x) - L^t_{a,b,c}y(x)| \leq \begin{cases} \frac{X}{c}\left(\frac{1}{c}-1\right)\frac{M}{12\Gamma(b)} \begin{cases} n^{-2}B\left(\frac{1}{c}+a-1, b\right), & \frac{1}{c}+a>1; \\ n^{-2}\ln n, & \frac{1}{c}+a=1; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(2-\frac{1}{c}-a\right), & \frac{1}{c}+a<1, \end{cases} & c \neq 1; \\ \frac{1}{n^2}\frac{X^2M}{12\Gamma(b)}B(a+1,b), & c=1. \end{cases} \tag{89}$$

$$\left|I_{a,b,c}y(x) - K^t_{a,b,c}y(x)\right| \leq \frac{1}{n^2}\frac{X^2}{12\Gamma(b)}MB(a+1,b). \tag{90}$$

To numerically illustrate the theorem, we conduct several simulations. First, we want to check the error constant in both Rectangular and Trapezoidal Rules. To this end, we choose $y(x) = x$ for the former and $y(x) = x$ or $y(x) = x^2/2$ for the latter. These functions are chosen to have a constant derivative regardless the point at which it is evaluated. Plots on Fig. 1 show an exemplary simulation confirming our theoretical results. Graphs represent the following ratio as a function of n different for each of the discretizations. For example,

$$\frac{\left|I_{a,b,c}y(x) - K^r_{a,b,c}y(x)\right|}{\frac{x}{2\Gamma(b)n}B(a+1,b)}, \tag{91}$$

It is clear that the above expressions (and its analogues) should approach 1 for large n.

The second verification we would like to conduct is the numerical calculation of particular orders of convergence. In our simulations, we use the trial function $y(x) = \exp(x)$ having all its derivatives different than zero at $x = 0, 1$. Due to our previous

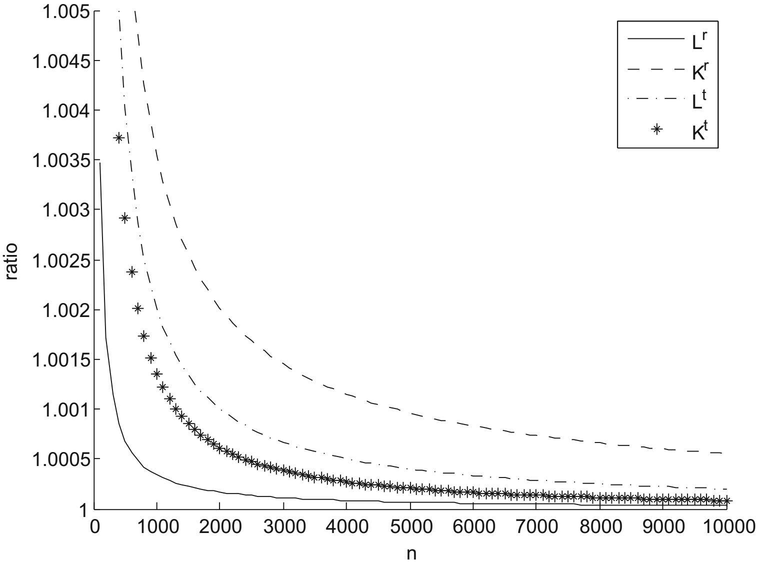

Fig. 1 Ratios of $|I_{a,b,c}y(x) - A_{a,b,c}y(x)|$ and its asymptotic limit as in (53)–(55) for $n \to \infty$. Here, $A_{a,b,c}$ is any operator from the legend and a choice of $y = y(x)$ is described in the text. Chosen parameters are $a = 0.5$, $b = 1.5$ and $c = 0.5$ and $x = 1$

remarks concerning a change of convergence rate for specific functions (which can make the kernel's singularity even weaker), we expect that numerical results will be in accord with Theorem 1. This is indeed the case and particular orders are given in Tables 1 and 2. The calculation was based on Aitken method based on Richardson extrapolation (see for ex. [32]), where the formula for order p is given by

$$p \approx \log_2 \frac{A_{a,b,c}\,(2n) - A_{a,b,c}\,(n)}{A_{a,b,c}\,(4n) - A_{a,b,c}\,(2n)}. \tag{92}$$

Here, $A_{a,b,c}\,(N)$ is one of the considered discretizations calculated for the number of grid points equal to N. We can see that in any case the numerical results confirm results of Theorem 1.

Table 1 Estimated orders of discretization error for the operator $L_{a,b,c}$

	$a = 1, b = 1.5, c = 0.5$	$a = -0.9, b = 0.5, c = 2$
Rectangular rule	1.0000	0.6050
Trapezoidal rule	2.0002	0.6000
Midpoint rule	1.9934	0.6008

Table 2 Estimated orders of discretization error for the operator $K_{a,b,c}$

	$a = 1, b = 1.5, c = 0.5$	$a = -0.9, b = 0.5, c = 2$
Rectangular rule	1.0001	1.0088
Trapezoidal rule	2.0001	1.9977
Midpoint rule	1.9985	1.1955

3 Finite difference scheme

In this section, we consider the following integro-differential equation with E-K operator

$$y' = f(x, y, I_{a,b,c}y), \quad y(0) = y_0, \quad x \in (0, X). \tag{93}$$

Normally, we would assume that f is Lipschitz continuous with respect to the second and third argument, i.e.

$$|f(x, u, p) - f(x, v, p)| \le L_1|u - v|, \quad |f(x, u, p) - f(x, u, q)| \le L_2|p - q|, \tag{94}$$

for some constants $L_{1,2} > 0$. Due to the existence theorems [22, 52], this is a natural assumption yielding the solution y to be at least once differentiable. However, to prove our results, we will have to assume a somewhat stronger regularity condition on f, namely

$$f \in C^2(\mathbb{R}^3), \tag{95}$$

which will be needed for using estimates on the trapezoidal quadrature's discretization error. We can see that (95) implies (94). In order to find a numerical solution of (93), we propose the following finite difference scheme which encompasses trapezoidal rules for both discretization of $I_{a,b,c}$ and the integro-differential equation itself

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n, K^t_{a,b,c}y_n) + f(x_{n+1}, y_{n+1}, K^t_{a,b,c}y_{n+1})\right), \tag{96}$$

where the discretization operator $K^t_{a,b,c}$ was defined in (23). We have the uniform grid $x_n = nh$, where $h = X/N$ for some $0 \le x \le X$ and $N \in \mathbb{N}$. Moreover, the numerical approximations are defined as usual $y_n \approx y(x_n)$, where y is the solution of (93).

In Theorem 1, we have shown that the discretization operator is of second order and thus we choose the same order of numerical scheme (trapezoidal for improved stability). Below, we show that indeed, our proposed scheme (96) is of second order. First, a result concerning the truncation error.

Theorem 2 (Truncation error) *Assume that f satisfies (95) for $x \in [0, X]$. Then for $a > -1$ and $b > 0$ and $c > 0$ the local truncation error for the finite difference scheme (96) is $O(h^3)$ as $h \to 0$.*

Proof We define a local truncation error e_{n+1} as the difference between the real solution $y(x_{n+1})$ of (93) and the approximation y_{n+1} provided that $y(x_i) = y_i$ for $i = 0, 1, \ldots, n$. Let us write then

$$e_{n+1} = y(x_{n+1}) - y_{n+1}.$$

Now, integrating both sides of the equation (93) from x_n to x_{n+1} and rearranging the terms gives a formula for $y(x_{n+1})$

$$e_{n+1} = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x), Iy(x))dx - \left(y_n + \frac{h}{2}\left(f(x_n, y_n, Ky_n)\right.\right.$$
$$\left.\left. + \; f(x_{n+1}, y_{n+1}, Ky_{n+1})\right)\right), \quad (97)$$

where we have suppressed writing a, b, c and the superscript t in order to keep the notation compact. Notice that the function f is locally bounded and, moreover, $\frac{d^2}{dx^2} f(x, y(x), I_{a,b,c}y(x))$ is also locally bounded by the assumption of f being C^2. This, along with $y(x_n) = y_n$ lets us use the trapezoidal rule for approximation of the integral

$$\begin{aligned} e_{n+1} = &\tfrac{h}{2}(f(x_n, y(x_n), Iy(x_n)) + f(x_{n+1}, y(x_{n+1}), Iy(x_{n+1}))) - Mh^3 \\ &-\tfrac{h}{2}\left(f(x_n, y_n, Ky_n) + f(x_{n+1}, y_{n+1}, Ky_{n+1})\right), \end{aligned} \quad (98)$$

where ξ_n is some point in the interval $[x_n, x_{n+1}]$ and M is a constant. Let us rearrange the terms of the above expression in a following manner

$$\begin{aligned} e_{n+1} &= \tfrac{h}{2}\left(f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Ky_n)\right) \\ &+\tfrac{h}{2}(f(x_{n+1}, y(x_{n+1}), Iy(x_{n+1})) - f(x_{n+1}, y_{n+1}, Ky_{n+1})) - Mh^3. \end{aligned} \quad (99)$$

From the Lipschitz condition, we know that there exist positive constants L_1 and L_2, such that

$$\begin{aligned} |e_{n+1}| \le &\tfrac{h}{2}L_1\left(|y(x_n) - y_n| + |y(x_{n+1}) - y_{n+1}|\right) + \tfrac{h}{2}L_2(|Iy(x_n) \\ &-Ky_n| + |Iy(x_{n+1}) - Ky_{n+1}|) + Mh^3. \end{aligned} \quad (100)$$

Again, we assumed that $y(x_i) = y_i$ for $i = 0, 1, \ldots, n$, so not only $|y(x_n) - y_n| = 0$, but also $Ky_n = Ky(x_n)$. Thus, (by the discretization error theorem (Theorem 1)), we know that there exists a positive constant D_1 such that $|Iy(x_n) - Ky_n| \le D_1h^2$. On the other hand, using the triangle inequality gives

$$|Iy(x_{n+1}) - Ky_{n+1}| \le |Iy(x_{n+1}) - Ky(x_{n+1})| + |Ky(x_{n+1}) - Ky_{n+1}| \le D_2h^2 + |Ke_{n+1}|, \quad (101)$$

for some positive constant D_2. Furthermore, from the definition (23) of the operator $K^t_{a,b,c}$, we know that

$$|Ke_{n+1}| = w_{n+1}|e_{n+1}|, \quad (102)$$

where w_{n+1} is the last weight in the trapezoidal scheme (23). Thus, we have obtained an upper bound for the right-hand side of (100)

$$\begin{aligned} |e_{n+1}| &\le \tfrac{h^3}{2}L_1D_1 + \tfrac{h}{2}L_2\left(|e_{n+1}| + D_2h^2 + w_{n+1}|e_{n+1}|\right) + \tfrac{h^3}{12}M = \\ &= \tfrac{h}{2}L_2\left(1 + w_{n+1}\right)|e_{n+1}| + h^3\left(\tfrac{L_1D_1+L_2D_2}{2} + \tfrac{M}{12}\right). \end{aligned} \quad (103)$$

Let us rewrite the above inequality in a following way

$$|e_{n+1}| \leq \frac{h^3 D_3}{1 - hD_4(1 + |w_{n+1}|)}, \tag{104}$$

where $D_3 = \left(\frac{L_1 D_1 + L_2 D_2}{2} + \frac{M}{12}\right)$ and $D_4 = \frac{L_2}{2}$ are positive. Since the step h can be arbitrary small, the denominator of the above expression goes to 1 when $h \to 0$, hence is $O(1)$. Thus, we have shown that

$$|e_{n+1}| = O(h^3), \text{ as} h \to 0. \tag{105}$$

□

Actually, a stronger result is true—the numerical scheme (96) is second-order convergent to the exact solution of the integro-differential equation (93). Here, we only prove convergence for $b \geq 1$ and leave the case $0 < b < 1$ for future work.

Theorem 3 (Convergence) *Assume that f satisfies (95) for $x \in [0, X]$. Then for $a > -1$ and $b \geq 1$ and $c > 0$, the finite difference scheme (96) is second-order convergent, i.e.*

$$|y(x_n) - y_n| = O(h^2) \quad as \quad h \to 0 \quad with \quad nh = const. \tag{106}$$

Proof Again, we start by introducing the standard notation for an error, namely

$$e_n := y(x_n) - y_n, \tag{107}$$

which is a difference between the exact solution evaluated at x_n and its approximation obtained by numerical iteration scheme (96). We can also assume that the initial values are exactly the same, i.e. $e_0 = 0$. Integrating (93) over the interval $[x_n, x_{n+1}]$ allows us to write

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x), Iy(x))dx \\ &= y(x_n) + \frac{h}{2}\left(f(x_n, y(x_n), Iy(x_n) + f(x_{n+1}, y(x_{n+1}), Iy(x_{n+1})\right) + Ch^3, \end{aligned} \tag{108}$$

where the last equality comes from the estimate for trapezoid quadrature (and C is the error constant) and we suppress writing a, b, c. Now, taking the difference with (96) gives

$$\begin{aligned} e_{n+1} &= e_n + \frac{h}{2}\left(f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Ky_n)\right) \\ &\quad + \frac{h}{2}\left(f(x_{n+1}, y(x_{n+1}), I_{a,b,c}y(x_{n+1})) - f(x_{n+1}, y_{n+1}, Ky_{n+1})\right) + Ch^3. \end{aligned} \tag{109}$$

A simple estimate for f leads to

$$\begin{aligned} &|f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Ky_n)| \leq |f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Iy(x_n))| \\ &+ |f(x_n, y_n, Iy(x_n)) - f(x_n, y_n, Ky(x_n)| + |f(x_n, y(x_n), Ky(x_n)) - f(x_n, y_n, Ky_n)| \end{aligned} \tag{110}$$

and a similar expression for the $n+1$ step. We can further estimate with a help of the Lipschitz condition (94) to obtain

$$|f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Ky_n)| \leq L_1 |e_n| + L_2 (|Iy(x_n) - Ky(x_n)| + K|e_n|), \quad (111)$$

where

$$K|e_n| = \sum_{i=1}^{n} w_i^t |e_i|, \quad (112)$$

since $e_0 = 0$. Here, we can think of y as being a piecewise constant function. Next, by the discretization error theorem (Theorem 1), we have

$$|Iy(x_n) - Ky(x_n)| \leq Dh^2, \quad (113)$$

where D is some (known) constant. Moreover, recalling the definition of weights w^t and using our assumption that $b \geq 1$, we can easily show

$$|w| \leq Wh, \quad 1 \leq i \leq n, \quad (114)$$

for some constant W. Putting these estimates together, we obtain

$$|f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Ky_n)| \leq L_1 |e_n| + L_2 \left(Dh^2 + Wh \sum_{i=1}^{n} |e_i| \right), \quad (115)$$

and similarly for $n+1$ step (where, after possible redefinition, we can retain the same constants). Now, we can go back to (109) and write

$$|e_{n+1}| \leq \left(1 + L_1 \frac{h}{2}\right) |e_n| + WL_2 h^2 \sum_{i=1}^{n} |e_i| + \left(L_1 \frac{h}{2} + WL_2 \frac{h^2}{2}\right) |e_{n+1}| + (DL_2 + C)h^3, \quad (116)$$

which implies

$$\begin{aligned} |e_{n+1}| &\leq \frac{1}{1 - L_1 \frac{h}{2} - WL_2 \frac{h^2}{2}} \left[\left(1 + L_1 \frac{h}{2}\right) |e_n| + WL_2 h^2 \sum_{i=1}^{n} |e_i| + (DL_2 + C)h^3 \right] \\ &\leq (1 + C_1 h) |e_n| + C_2 h^2 \sum_{i=1}^{n} |e_i| + C_3 h^3, \end{aligned} \quad (117)$$

where $C_{1,2,3} > 0$ are some constants. The last inequality follows from the fact that $\left(1 - L_1 \frac{h}{2} - WL_2 \frac{h^2}{2}\right) = O(1)$. We have thus obtained a recurrence relation for the error $|e_{n+1}|$. It is possible to derive a Gronwall-type estimate on such a given expression (see [4]). To this end, we claim that

$$|e_{n+1}| \leq \frac{\left(1 + C_1 h + C_2 (n+1) h^2\right)^{n+1} - 1}{C_1 h} C_3 h^3. \quad (118)$$

From this, we easily can obtain the assertion by noting that $(n+1)h$ is fixed and thus there exists a constant C_4 such that $(1+C_1h+C_2(n+1)h^{1+\delta})^{n+1} \leq (1+C_4h)^{n+1} \leq e^{C_4}$. This implies

$$|e_{n+1}| \leq \frac{e^{C_4}-1}{C_1h}C_3h^3 = O(h^2). \tag{119}$$

Henceforth, we are left with demonstrating that (117) forces (118). Proceed by induction and from (117) verify that for $n=0$, we have

$$|e_1| \leq C_3h^3 \leq \left(1+\frac{C_2}{C_1}h\right)C_3h^3 \leq \frac{\left(1+C_1h+C_2h^2\right)-1}{C_1h}C_3h^3. \tag{120}$$

Now, assume that (117) holds for $1, 2, ..., n$. Then

$$\begin{aligned} |e_{n+1}| &\leq (1+C_1h)\,\tfrac{\left(1+C_1h+C_2nh^2\right)^n-1}{C_1h}C_3h^3 + C_2h^2\textstyle\sum_{i=1}^n \tfrac{\left(1+C_1h+C_2ih^2\right)^i-1}{C_1h}C_3h^3 + C_3h^3 \\ &\leq \tfrac{1}{C_1h}\left[(1+C_1h)\left(\left(1+C_1h+C_2nh^2\right)^n-1\right)+C_2h^2\left(1+C_1h+C_2nh^2\right)^n+C_1h\right]C^3h^3 \\ &= \tfrac{1}{C_1h}\left[\left(1+C_1h+C_2nh^2\right)^{n+1}-1\right]C^3h^3 \leq \tfrac{1}{C_1h}\left[\left(1+C_1h+C_2(n+1)h^2\right)^{n+1}-1\right]C^3h^3. \end{aligned} \tag{121}$$

Induction is complete and this concludes the proof. □

To make an illustration of the theoretical results, we conduct numerical calculations. The order of convergence can be verified by studying an equation with a known

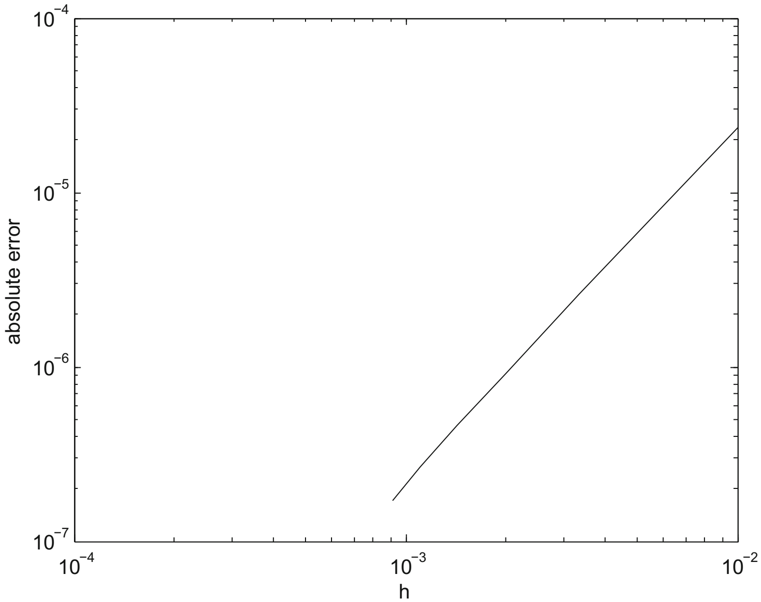

Fig. 2 Difference between exact and numerical solutions of the equation (122) for different values of h computed at $x = 1$

exact solution. This, for example, is

$$y' = I_{0,1,1}y - \frac{1}{4}x^2 + 2x, \quad y(0) = 0, \tag{122}$$

which has an exact solution $y(x) = x^2$. As a check of the global error of convergence, we compare the values of x^2 and its numerical approximations being the solutions of the above equation for different number of steps. As a point at which the error is calculated, we take $x = 1$ and plot the error in a log-log scale (Fig. 2). As it can be seen from the plot, the rate of convergence is equal to 2 which is exactly stated by the above theorem.

4 Conclusion

Equations involving Erdélyi-Kober fractional operator are starting to emerge in some fields of mathematics and physics. In particular, time-fractional porous medium equation transforms into a self-similar form having this kind of nonlocal operator. Motivated by that example we have proposed a discretization method and a second-order finite difference scheme to solve integro-differential equations with E-K operator. Asymptotic forms of the discretization errors have been found for some variants of numerical schemes. The scope of our future work includes finding the complete asymptotic series for errors (generalization of Theorem 1) and applying our results to the time-fractional porous medium equation. For the latter, the issue of E-K kernel integrability is very subtle.

Acknowledgments This research was supported by the National Science Centre, Poland under the project with a signature NCN $2015/17/D/ST1/00625$.

References

1. Bader, A.-S., Kiryakova, V.S.: Explicit solutions of fractional integral and differential equations involving Erdélyi-Kober operators. Appl. Math. Comput. **95**(1), 1–13 (1998)
2. Atkinson, K.E.: The numerical solution of an Abel integral equation by a product trapezoidal method. SIAM J. Numer. Anal. **11**(1), 97–101 (1974)
3. Awotunde, A.A., et al.: Numerical schemes for anomalous diffusion of single-phase fluids in porous media. Commun. Nonlinear Sci. Numer. Simul. **39**, 381–395 (2016)
4. Baker, C.T.H.: A perspective on the numerical treatment of Volterra equations. J. Comput. Appl. Math. **125**(1), 217–249 (2000)
5. Baleanu, D., Güvenç, Z.B., Machado, J.T.: New trends in nanotechnology and fractional calculus applications. Springer (2010)
6. Baleanu, D., et al.: Models and numerical methods. World Sci. **3**, 10–16 (2012)
7. Bronstein, I., et al.: Transient anomalous diffusion of telomeres in the nucleus of mammalian cells. Phys. Rev. Lett. **103**(1), 018102 (2009)

8. Brunner, H., Houwen, P.J.: The numerical solution of Volterra equations, vol. 3. Elsevier Science Ltd (1986)
9. Buckwar, E., Luchko, Y.: Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. J. Math. Anal. Appl. **227**(1), 81–97 (1998)
10. Chen, C., Jiang, Y.-L.: Lie group analysis method for two classes of fractional partial differential equations. Commun. Nonlinear Sci. Numer. Simul. **26**(1), 24–35 (2015)
11. Chuvilgin, L.G., Ptuskin, V.S.: Anomalous diffusion of cosmic rays across the magnetic field. Astron. Astrophys. **279**, 278–297 (1993)
12. Costa, F.S., et al.: Similarity solution to fractional nonlinear space-time diffusion-wave equation. J. Math. Phys. **56**(3), 033507 (2015)
13. Demir, A., Kanca, F., Ozbilge, E.: Numerical solution and distinguishability in time fractional parabolic equation. Bound. Value Probl. **2015**(1), 1 (2015)
14. El Abd, A.: A method for moisture measurement in porous media based on epithermal neutron scattering. Appl. Radiat. Isot. **105**, 150–157 (2015)
15. Erdélyi, A.: On fractional integration and its application to the theory of Hankel transforms. Q. J. Math. **1**, 293–303 (1940)
16. Ford, N.J., Simpson, A.C.: The numerical solution of fractional differential equations: speed versus accuracy. Numer. Algorithm. **26**(4), 333–346 (2001)
17. Gazizov, R.K., Ibragimov, N.H., Lukashchuk, S.Y.: Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations. Commun. Nonlinear Sci. Numer. Simul. **23** (2015)
18. Gazizov, R.K., Kasatkin, A.A., Lukashchuk, S.Y.: Symmetry properties of fractional diffusion equations. In: Physica Scripta 2009. T136, p. 014016 (2009)
19. Gorenflo, R., Luchko, Y., Mainardi, F.: Wright functions as scale-invariant solutions of the diffusion-wave equation. J. Comput. Appl. Math. **118** (2000)
20. Herrmann, R.: Towards a geometric interpretation of generalized fractional integrals—Erdélyi-Kober type integrals on R N, as an example. Fractional Calc. Appl. Anal. **17** (2014)
21. Hilfer, R.: Applications of fractional calculus in physics. World Scientific (2000)
22. Ibrahim, R.W., Momani, S.: On the existence and uniqueness of solutions of a class of fractional differential equations. J. Math. Anal. Appl. **334**(1) (2007)
23. Kepten, E.: Uniform contraction-expansion description of relative centromere and telomere motion. Biophys. J. **109**(7) (2015)
24. Kiryakova, V., Al-Saqabi, B.: Explicit solutions to hyper-Bessel integral equations of second kind. Comput. Math. Appl. **37**(1) (1999)
25. Kiryakova, V.S.: Generalized fractional calculus and applications. CRC Press (1993)
26. Kiryakova, V.S., Al-Saqabi, B.N.: Transmutation method for solving Erdélyi–Kober fractional differintegral equations. J. Math. Anal. Appl. **1997**(1)
27. Kober, H.: On fractional integrals and derivatives. Q. J. Math. **11** (1940)
28. Küntz, M., Lavallée, P.: Experimental evidence and theoretical analysis of anomalous diffusion during water infiltration in porous building materials. J. Phys. D. Appl. Phys. **34**(16) (2001)
29. Li, C., Yi, Q., Chen, A.: Finite difference methods with non-uniform meshes for nonlinear fractional differential equations. J. Comput. Phys. **316** (2016)
30. Li, X., Xu, C.: A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anal. **47**(3) (2009)
31. Lin, Y., Xu, C.: Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. **225**(2) (2007)
32. Linz, P.: Analytical and numerical methods for Volterra equations, vol. 7. Siam (1985)
33. Luchko, Y.F., Srivastava, H.M.: The exact solution of certain differential equations of fractional order by using operational calculus. Comput. Math. Appl. **29**(8) (1995)
34. Luchko, Y.: Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. Fractional Calc. Appl. Anal. **15**(1) (2012)
35. Lyness, J., Ninham, B.W.: Numerical quadrature and asymptotic expansions. Math. Comput. **21**(98) (1967)
36. Mainardi, F., Luchko, Y., Pagnini, G.: The fundamental solution of the space-time fractional diffusion equation. In: arXiv preprint arXiv:0702419 (2007)
37. Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**(1) (2000)

38. Miller, K.S., Ross, B.: An introduction to the fractional calculus and fractional differential equations (1993)
39. Pablo, A., et al.: A fractional porous medium equation. Adv. Math. **226**(2) (2011)
40. Pachepsky, Y., Timlin, D., Rawls, W.: Generalized Richards' equation to simulate water transport in unsaturated soils. J. Hydrol. **272**(1) (2003)
41. Pagnini, G.: Erdélyi-Kober fractional diffusion. Fractional Calc. Appl. Anal. **15**(1) (2012)
42. Płociniczak, Ł.: Analytical studies of a time-fractional porous medium equation. Derivation, approximation and applications. Commun. Nonlinear Sci. Numer. Simul. **24**(1), 169–183 (2015)
43. Płociniczak, Ł.: Approximation of the Erdélyi–kober operator with application to the time-fractional porous medium equation. SIAM J. Appl. Math. **74**(4), 1219–1237 (2014)
44. Płociniczak, Ł.: Diffusivity identification in a nonlinear time-fractional diffusion equation. Fractional Calc. Appl. Anal. **19**(4), 843–866 (2016)
45. Płociniczak, Ł., Okrasińska, H.: Approximate self-similar solutions to a nonlinear diffusion equation with time-fractional derivative. Physica D: Nonlinear Phenomena **261**, 85–91 (2013)
46. Sabatier, J., Agrawal, O.P., Tenreiro Machado, J.A.: Advances in fractional calculus, vol. 4, p. 9. Springer (2007)
47. Sahadevan, R., Bakkyaraj, T.: Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations. J. Math. Anal. Appl. **393**(2), 341–347 (2012)
48. Sahadevan, R., Bakkyaraj, T.: Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. Fractional Calc. Appl. Anal. **18**(1), 146–162 (2015)
49. Santos-León, J.C.: Asymptotic expansions for trapezoidal type product integration rules. J. Comput. Appl. Math. **91**(2), 219–230 (1998)
50. Sneddon, I.N.: The use in mathematical physics of Erdelyi-Kober operators and of some of their generalizations. In: Fractional Calculus and its applications, pp. 37–79. Springer (1975)
51. Sun, H.G., et al.: A fractal Richards equation to capture the non-Boltzmann scaling of water transport in unsaturated media. Adv. Water Resour. **52**, 292–295 (2013)
52. Wang, J.R., Dong, X.W., Zhou, Y.: Analysis of nonlinear integral equations with Erdélyi–Kober fractional operator. Commun. Nonlinear Sci. Numer. Simul. **17**(8), 3129–3139 (2012)
53. Weiss, M., Hashimoto, H., Nilsson, T.: Anomalous protein diffusion in living cells as seen by fluorescence correlation spectroscopy. Biophys. J. **84**(6), 4043–4052 (2003)
54. Weiss, R.: Product integration for the generalized Abel equation. Math. Comput. **26**(117), 177–190 (1972)
55. Zhokh, A.A., Trypolskyi, A.I., Strizhak, P.E.: An investigation of anomalous time-fractional diffusion of isopropyl alcohol in mesoporous silica. Int. J. Heat Mass Trans. **104**, 493–502 (2017)